Waiting Times and Convergence of a Spatially Discrete Thin Film Equation

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Abstract

We consider the fourth order thin film equation. Firstly, we introduce a scheme for a spatial discretization of this equation. To this aim, we discuss how to rewrite the equation using the Lagrangian picture. It allows us to define a spatial discretization based on "mass coordinates". We discuss the convergence of discrete solutions to the strong solution of the initial equation. We then turn to the question of the dynamical properties of another discretization of the initial equation. The waiting time phenomenon of the edge of the support is introduced and then treated following the ideas of [1].

The equation

Consider the fourth order nonlinear degenerate parabolic equation

$$u_t + (u^n u_{xxx})_x = 0 \quad \text{with } t > 0,\ x \in \Omega, \tag{1}$$

of the function $u(x,t)$ where $\Omega = [a,b]$ for some $a, b \in \mathbb{R}$, $a < b$. The parameter $n \in \mathbb{R}$ is such that $n \in (0,2) \setminus \{1\}$.

The boundary conditions are

$$u_x = u^n u_{xxx} = 0 \quad \text{for } x \in \partial\text{supp}(u) \quad \text{for every } t > 0.$$

The initial value is given by a function $u_0 \in H^1(\Omega)$:

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

and $u_0 \geq 0$ almost everywhere in Ω. We assume that the total mass of $u_0(x)$ is one:

$$\int_\Omega u_0(x)\mathrm{d}x = 1. \tag{2}$$

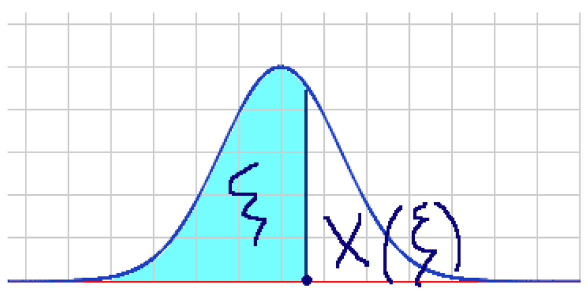
Lagrangian picture

The total mass (2) is preserved, that is,

$$\int_\Omega u(x,t)\mathrm{d}x = 1 \text{ for all } t > 0.$$

The "mass coordinates" are maps $X(\cdot, t) : [0,1] \to \mathbb{R}$ implicitly given by

$$\xi = \int_{-\infty}^{X(\xi,t)} u(x,t)\mathrm{d}x \text{ for all } \xi \in [0,1].$$



Consequently, for each $\xi \in [0,1]$ the amount of mass left to $X(\xi,t)$ equals ξ, and the new coordinates trace the masses.

We introduce the new variable

$$Z(\xi,t) = \frac{1}{X_\xi(\xi,t)} = u(X(\xi,t),t)$$

and want to rewrite the equation (1) in terms of the variables X and Z. Two equivalent formulations are

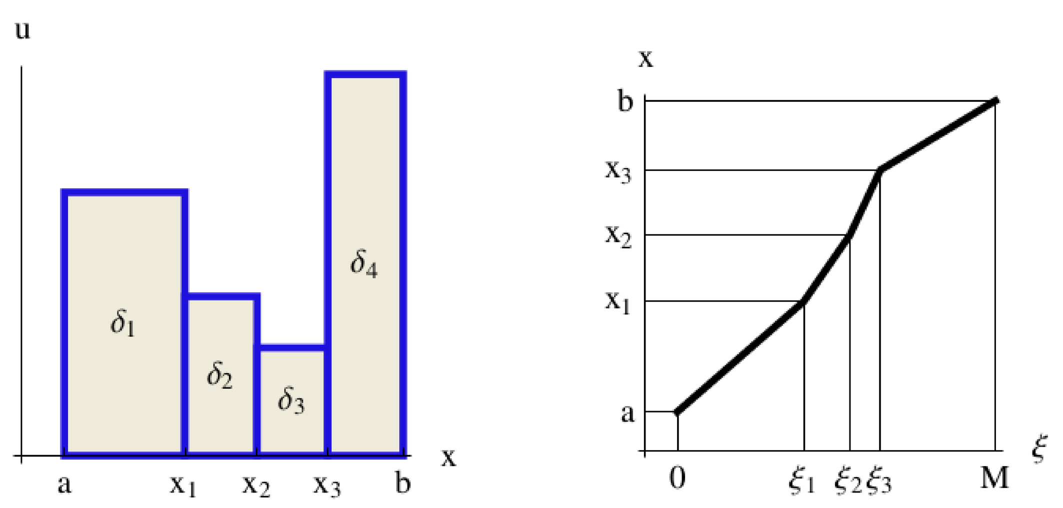
$$\partial_t X = Z^n\big(Z(ZZ_\xi)_\xi\big)_\xi, \tag{3}$$

$$\partial_t X = Z^{n-1}\Big(Z^3 Z_{\xi\xi} + \frac{1}{2}Z^2 Z_\xi^2\Big)_\xi. \tag{4}$$

Spatial discretizations

We discretize first with respect to the mass coordinate $\xi \in [0,1]$. The discretization parameter is $K \geq 2$, and $[0,1]$ is subdivided into K equidistant intervals of the length δ. This mesh is $\xi_k, k \in \{0,1,2,...,K\}$.

The corresponding points $x_k \in \mathbb{R}$ and values $u_\kappa = z_\kappa > 0, \kappa \in \{\frac{1}{2}, \frac{3}{2}, ..., K - \frac{1}{2}\}$ are such that $z_\kappa = \frac{\delta}{x_{\kappa+1/2} - x_{\kappa-1/2}}$.



The correspondence for a non-equidistant discretization of the mass space. Source: [2].

The equations (3) and (4) motivate us to introduce the discretizations

$$\dot{x}_k = \mathrm{R}_k \mathrm{D}_k\Big(z\mathrm{D}^*\Big(\frac{\mathrm{D}z}{\mathrm{M}(z^{-1})}\Big)\Big), \text{ where } \mathrm{R}_k = (2-n)\frac{\mathrm{D}_k z}{\mathrm{M}_k(z^{-1})\mathrm{D}_k(z^{2-n})}, \tag{5}$$

$$\dot{x}_k = \frac{(2-n)\mathrm{D}_k z}{\mathrm{D}_k(z^{2-n})}\mathrm{D}_k(z^3\Delta z + \frac{1}{2}z^2\Delta(z^2)). \tag{6}$$

Here D and D^* are the usual discrete difference operators corresponding to the integer or half-integer indices, Δ is the discrete laplacian and M is the mean value. For details see [1].

Convergence of the discrete scheme

Consider the spatial discretizations (5) with the assumption that there exist constants $z_{min} > 0$ and $0 < z_{max} < \infty$ such that $z_{min} \leq z_\kappa \leq z_{max}$ for all κ.

Construct the discrete density $\overline{u}^\delta : \Omega \to [0,\infty)$ as $\overline{u}^\delta(x) = z_\kappa = u_\kappa$ when $x \in [x_{\kappa-1/2}, x_{\kappa+1/2})$.

Fix a time horizon $T > 0$.

Theorem. There exists $u_* \in L^2(0,T;H^2(\Omega))$ such that up to a subsequence $\overline{u}^\delta \to u_*$ in $L^2(0,T;H^2(\Omega))$ as $\delta \to 0$.

The limit curve $u_* \in L^2(0,T;H^2(\Omega))$ is the strong solution of (1).

A strong solution must satisfy for all $\varphi \in C_0^\infty(\Omega)$, $\psi \in C_0^\infty((0,T))$ the equation

$$\iint_Q u(x,t)\varphi(x)\psi'(t)\mathrm{d}t\mathrm{d}x = -\iint_Q (u^n(x,t)\varphi'(x))_x u_{xx}(x,t)\psi(t)\mathrm{d}t\mathrm{d}x,$$

where $Q = (0,T) \times \Omega$.

Waiting times

Consider the discretizations (6) of the equation (1).

The waiting time phemomenon means that the support of the solution cannot expand immediately after initialization. The edge of support only moves when the solution has gained a certain steepness there, see the condition (7) below.

Theorem. Provided that

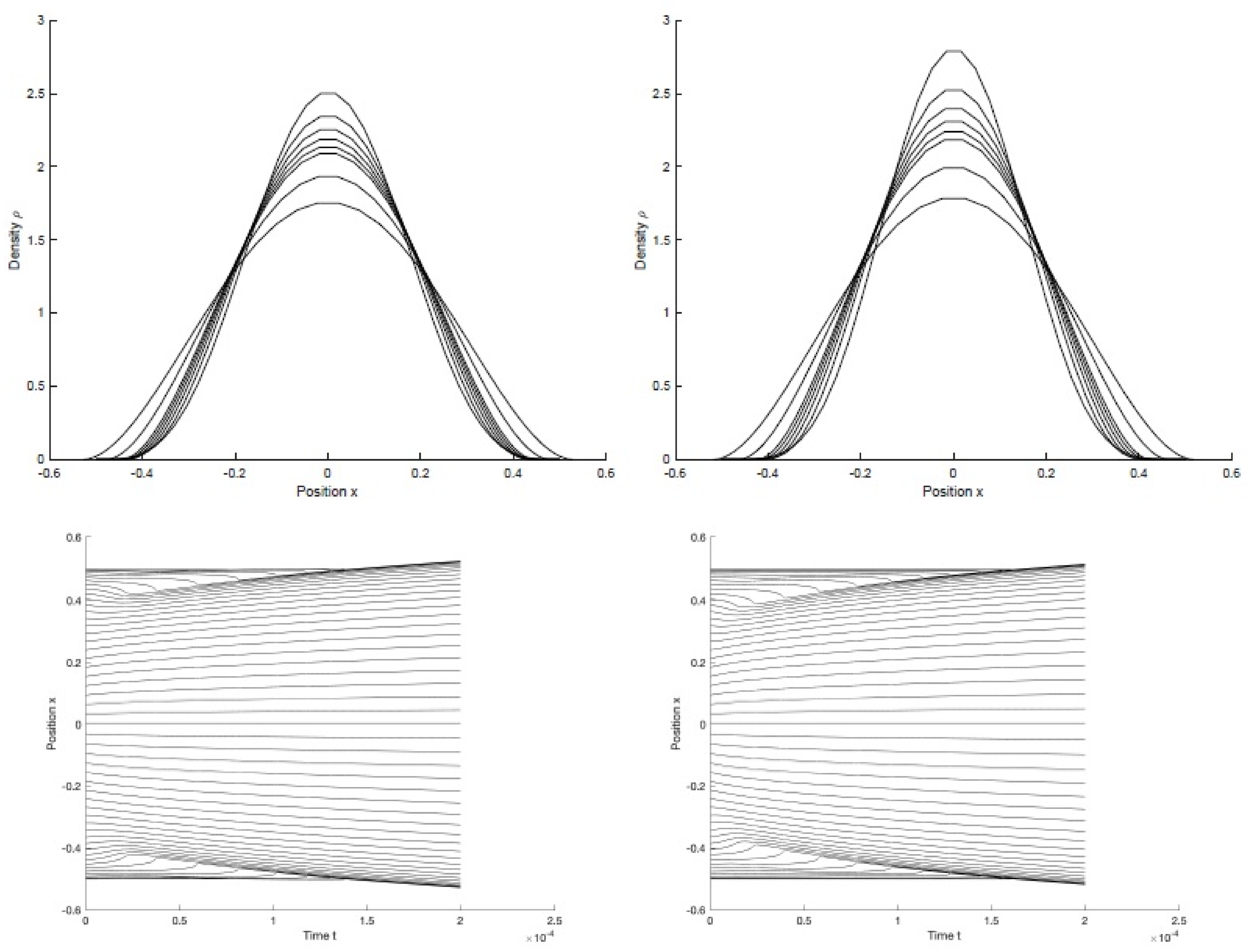
$$\overline{b} = \frac{1}{1-n}\sup_{l\in(a,b)} \frac{\int_{\overline{x}_0}^l u_0(x)^{2-n}\mathrm{d}x}{\left(\int_{\overline{x}_0}^l u_0(x)\mathrm{d}x\right)^{\frac{8-3n}{4+n}}} < \infty, \tag{7}$$

where $\overline{x}_0 = x_0(0)$, then a waiting time occurs at the left edge of support: there exists $T > 0$ such that

$$|x_0(t^*) - \overline{x}_0| \leq C\delta^{\frac{n}{4+n}}(t^*)^{\frac{2-n}{5}}\overline{b}^{\frac{3+n}{5}}$$

for all the times $t^* < T$.

We provide a qualitative illustration of the solutions to the discrete thin film equation with $n = 1$ and $K = 50$. The top row shows an overlay of snapshots of the density in physical space at different instances of time, the bottom row shows the position of the Lagrangian points $x_k(t)$ as functions of time. The two columns correspond to two particular choices of the initial condition $u_0(x)$.



Source: [1].

Selected references

[1] J. Fischer and D. Matthes, *The waiting time phenomenon in spatially discretized porous medium and thin film equations*, SIAM J. Math. Anal. (2021), pp. 59(1):60-87.

[2] D. Matthes and H. Osberger, *Convergence of a variational Lagrangian scheme for a nonlinear drift diffusion equation*, ESAIM: M2AN, vo. 48, no. 3 (2014), pp. 697-726.